

Statements and Dilemmas Regarding the ℓ^2 -homology of Coxeter groups

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Abstract

We generalize the methods used in [11] to provide a program for proving Singer's Conjecture for Coxeter systems. Specifically, we consider *even* Coxeter systems with nerves that are flag triangulations of \mathbb{S}^{n-1} , $n = 2k$. We prove that Conjecture 1.3 in dimensions $n - 2$ and $n - 1$, along with the vanishing of the ℓ^2 -homology of certain subspaces called "two-letter" ruins above dimension $k + 1$, imply Conjecture 1.3 in dimension n . This is but a program. The author intends this paper to serve as a reference for those inquiring about Singer's Conjecture and about even Coxeter systems. Users of this paper should focus attention on Sections 4.4 and 4.5, along with Remark 4.5.2.

1 Introduction

The following conjecture is attributed to Singer.

Singer's Conjecture 1.1. *If M^n is a closed aspherical manifold, then the reduced ℓ^2 -homology of M^n , $\mathcal{H}_*(\widetilde{M}^n)$, vanishes for all $*$ $\neq \frac{n}{2}$.*

For details on ℓ^2 -homology theory, see [6], [7] and [8]. Now, let X be a geometric G -complex. A key feature of the ℓ^2 -theory is that it is possible to attach to the Hilbert space $\mathcal{H}_i(X)$ a nonnegative real number, called the i^{th} ℓ^2 -Betti number. A formula of Atiyah states that the alternating sum of these ℓ^2 -Betti numbers is the orbihedral Euler characteristic $\chi^{\text{orb}}(X/G)$, or in the case of a free action, the ordinary Euler characteristic $\chi(X/G)$. Thus, Conjecture 1.1 implies the following conjecture regarding Euler characteristic (attributed to H.Hopf):

The Euler Characteristic Conjecture 1.2. *If M^{2k} is a closed, aspherical manifold of dimension $2k$, then its Euler characteristic, $\chi(M^{2k})$, satisfies*

$$(-1)^k \chi(M^{2k}) \geq 0.$$

Singer's conjecture holds for elementary reasons in dimensions ≤ 2 . Indeed, top-dimensional cycles on manifolds are constant on each component, so a square-summable cycle on an infinite component is constant 0. As a result, Conjecture 1.1 in dimension ≤ 2 follows from Poincaré duality. In [9], Lott and Lück prove that it holds for those aspherical 3-manifolds for which Thurston's Geometrization Conjecture is true. (Hence, by Perelman, all aspherical 3-manifolds.)

Let S be a finite set of generators. A *Coxeter matrix* on S is a symmetric $S \times S$ matrix $M = (m_{st})$ with entries in $\mathbb{N} \cup \{\infty\}$ such that each diagonal entry is 1 and each off diagonal entry is ≥ 2 . The matrix M gives a presentation for an associated *Coxeter group* W :

$$W = \langle S \mid (st)^{m_{st}} = 1, \text{ for each pair } (s, t) \text{ with } m_{st} \neq \infty \rangle. \quad (1.1)$$

The pair (W, S) is called a *Coxeter system*. Denote by L the nerve of (W, S) . In several papers (e.g., [3], [4], and [6]), M. Davis describes a construction which associates to any Coxeter system (W, S) , a simplicial complex $\Sigma(W, S)$, or simply Σ when the Coxeter system is clear, on which W acts properly and cocompactly. The two salient features of Σ are that (1) it is contractible and (2) it permits a cellulation under which the link of each vertex is L . It follows that if L is a triangulation of \mathbb{S}^{n-1} , Σ is an n -manifold. There is a special case of Singer's conjecture for such manifolds.

Singer's Conjecture for Coxeter groups 1.3. *Let (W, S) be a Coxeter system such that its nerve, L , is a triangulation of \mathbb{S}^{n-1} . Then*

$$\mathcal{H}_i(\Sigma(W, S)) = 0 \text{ for all } i \neq \frac{n}{2}.$$

In [7], Davis and Okun prove that if Conjecture 1.3 for *right-angled* Coxeter systems is true in some odd dimension n , then it is also true for right-angled systems in dimension $n + 1$. (A Coxeter system is right-angled if generators either commute or have no relation.) They also show that Thurston's Geometrization Conjecture holds for these Davis 3-manifolds arising from right-angled Coxeter systems. Hence, the Lott and Lück result implies that Conjecture 1.3 for right-angled Coxeter systems is true for $n = 3$ and, therefore, also for $n = 4$. (Davis and Okun also show that Andreev's theorem, [1, Theorem 2], implies Conjecture 1.3 in dimension 3 for right-angled systems.) In [12], the author geometrizes arbitrary 3-dimensional Davis manifolds and shows that Conjecture 1.3 in dimension 3 follows.

Right-angled Coxeter systems are specific examples of *even* Coxeter systems. We say a Coxeter system is even if for any two generators $s \neq t$, m_{st} is either even or infinite. In [11], the author proves the following extension of the Davis-Okun 4-dimensional result:

Theorem 1.4. *Let (W, S) be an even Coxeter system whose nerve L is a flag triangulation of \mathbb{S}^3 . Then $\mathcal{H}_i(\Sigma(W, S)) = 0$ for $i \neq 2$.*

The purpose of this paper is to generalize the methods used in [11] to any dimension. Following that template, we look at specific subspaces Ω of Σ called *ruins* (see 2.2). What follows is a similar, but much more complicated, statement to that proven by Davis and Okun in [7]. For $n = 2k$ we consider even Coxeter systems with flag nerves. We prove that Conjecture 1.3 in dimensions $n - 2$ and $n - 1$, along with the vanishing of the ℓ^2 -homology of certain subspaces called “two-letter” ruins above dimension $k + 1$, implies Conjecture 1.3 in dimension n .

2 The Davis Complex

Let (W, S) be a Coxeter system. Denote by \mathcal{S} the poset of spherical subsets of S , partially ordered by inclusion; and let $\mathcal{S}^{(k)} := \{T \in \mathcal{S} \mid \text{Card}(T) = k\}$. Given a subset V of S , let $\mathcal{S}_{<V} := \{T \in \mathcal{S} \mid T \subset V\}$. Similar definitions exist for $>, \leq, \geq$. For any $w \in W$ and $T \in \mathcal{S}$, we call the coset wW_T a *spherical coset*. The poset of all spherical cosets we will denote by WS .

The poset $\mathcal{S}_{>\emptyset}$ is an abstract simplicial complex, denote it by L , and call it the *nerve* of (W, S) . The vertex set of L is S and a non-empty subset of vertices T spans a simplex of L if and only if T is spherical.

Let $K = |\mathcal{S}|$, the geometric realization of the poset \mathcal{S} . In K , simplices correspond to linearly ordered chains in the poset \mathcal{S} . It is the cone on the barycentric subdivision of L , the cone point corresponding to the empty set, and thus a finite simplicial complex. Denote by $\Sigma(W, S)$, or simply Σ when the system is clear, the geometric realization of the poset WS . This is the Davis complex. The natural action of W on WS induces a simplicial action of W on Σ which is proper and cocompact. K includes naturally into Σ via the map induced by $T \rightarrow W_T$, $T \in \mathcal{S}$. So we view K as a subcomplex of Σ and note that it is a strict fundamental domain for the action of W on Σ .

For any element $w \in W$, write wK for the w -translate of K in Σ . Let $w, w' \in W$ and consider $wK \cap w'K$. This intersection is non-empty if and only if $V = S(w^{-1}w')$ is a spherical subset. In fact, $wK \cap w'K$ is simplicially isomorphic to $|\mathcal{S}_{\geq V}|$, the geometric realization of $\mathcal{S}_{\geq V} := \{V' \in \mathcal{S} \mid V \subseteq V'\}$.

A cubical structure on Σ . For each $w \in W$, $T \in \mathcal{S}$, denote by $w\mathcal{S}_{\leq T}$ the subposet $\{wW_V \mid V \subseteq T\}$ of WS . Put $n = \text{Card}(T)$. $|w\mathcal{S}_{\leq T}|$ has the combinatorial structure of a subdivision of an n -cube. We identify the sub-simplicial complex $|w\mathcal{S}_{\leq T}|$ of Σ with this coarser cubical structure and call it a *cube of type T* . Note that the vertices of these cubes correspond to spherical subsets $V \in \mathcal{S}_{\leq T}$. (For details on this cubical structure, see [10].)

A cellulation of Σ by Coxeter cells. Σ has a coarser cell structure: its cellulation by “Coxeter cells.” (References for this cellulation include [4] and [7].) The features of the Coxeter cellulation are summarized by the following from [4].

Proposition 2.1. *There is a natural cell structure on Σ so that*

- *its vertex set is W , its 1-skeleton is the Cayley graph of (W, S) and its 2-skeleton is a Cayley 2-complex.*
- *each cell is a Coxeter cell.*
- *the link of each vertex is isomorphic to L (the nerve of (W, S)) and so if L is a triangulation of \mathbb{S}^{n-1} , Σ is a topological n -manifold.*
- *a subset of W is the vertex set of a cell if and only if it is a spherical coset and*
- *the poset of cells is WS .*

We will write Σ_{cc} , when necessary, to denote the Davis complex equipped with this cellulation by Coxeter cells. Under this cellulation, the vertices of Σ_{cc} correspond to cosets of W_\emptyset , i.e. to elements from W ; and 1-cells correspond to cosets of W_s , $s \in S$. It will be our convention to use the term “vertices” for vertices in the cellulation of Σ by Coxeter cells or for vertices in L and to use “0-simplices” for 0-simplices in K or translates of K .

2.2 Ruins

The following subspaces are defined in [5]. Let (W, S) be a Coxeter system. For any $U \subseteq S$, let $\mathcal{S}(U) = \{T \in \mathcal{S} \mid T \subseteq U\}$ and let $\Sigma(U)$ be the subcomplex of Σ_{cc} consisting of all cells of type T , with $T \in \mathcal{S}(U)$.

Given $T \in \mathcal{S}(U)$, define three subcomplexes of $\Sigma(U)$:

$\Omega(U, T)$: the union of closed cells of type T' , with $T' \in \mathcal{S}(U)_{\geq T}$,

$\hat{\Omega}(U, T)$: the union of closed cells of type T'' , $T'' \in \mathcal{S}(U)$, $T'' \notin \mathcal{S}(U)_{\geq T}$,

$\partial\Omega(U, T)$: the cells of $\Omega(U, T)$ of type T'' , with $T'' \notin \mathcal{S}(U)_{\geq T}$.

The pair $(\Omega(U, T), \partial\Omega(U, T))$ is called the (U, T) -ruin. For $T = \emptyset$, we have $\Omega(U, \emptyset) = \Sigma(U)$ and $\partial\Omega(U, \emptyset) = \emptyset$.

One-Letter Ruins. Let $t \in S$. We call the (S, t) -ruin a *one-letter ruin*. Put $U := \{s \in S \mid m_{st} < \infty\}$, i.e. U is the vertex set of the star of t in L . 1-cells in $\Omega(S, t)$ are of type u where $u \in U$. So two vertices w, v in a component of $\Omega(S, t)$, thought of as group elements of W , have the property that $v = wp$, where $p \in W_U$. Thus, the path components of $\Omega(S, t)$ are indexed by the cosets W/W_U . Denote by Ω the path-component of $\Omega(S, t)$ with vertex set corresponding W_U . The action of W_U on Σ restricts to an action on Ω . Put $K(U) := K \cap \Omega$ and note that the W_U -translates of $K(U)$ cover Ω , i.e. $\Omega = \bigcup_{w \in W_U} wK(U)$. Let $\partial\Omega := \Omega \cap \partial\Omega(S, t)$. Coxeter 1-cells of $\partial\Omega(S, t)$ are of type u where $u \in U - t$; so the path components of $\partial\Omega$ are indexed by the cosets W_U/W_{U-t} .

Boundary collars. If we restrict our attention to cubes of type T , where $T \subseteq T'$ for some $T' \in \mathcal{S}_{\geq t}$, Ω is a cubical complex and $\partial\Omega$ is a subcomplex. Moreover, if B is a component of $\partial\Omega$, the space $D := B \times [0, 1]$ is isomorphic to the union of the w -translates of $K(U)$ where w is a vertex of B . We call such subspaces *boundary collars*. It is clear that the collection of boundary collars covers Ω . We denote by $\partial_{in}(D)$ the end of this product which does not lie in $\partial\Omega$; the 0-simplices of $\partial_{in}(D)$ correspond to elements of $\mathcal{S}_{\geq t}$. The boundary collars intersect along subsets of these “inner” boundaries.

Two-Letter Ruins. For $U \subseteq S$ and $T \in \mathcal{S}(U)$ with $\text{Card}(T) = 2$, we call the (U, T) -ruin a “two-letter” ruin.

3 Variations on Singer’s Conjecture

In [7, Section 8], Davis and Okun present several variations of Singer’s Conjecture for Coxeter groups (Conjecture 1.3) in the case W is a right-angled Coxeter system. They then prove several implications regarding these statements including their proof that Conjecture 1.3 in dimension $2k - 1$ implies 1.3 in dimension $2k$. We proceed similarly, beginning with a restatement of Singer’s Conjecture. The Roman numeral notation is to model that used in [7] and [5].

I(n). *Suppose (W, S) is a Coxeter system such that its nerve, L , is a triangulation of \mathbb{S}^{n-1} . Then*

$$\mathcal{H}_i(\Sigma(W, S)) = 0 \text{ for } i \neq \frac{n}{2}.$$

I(1) and I(2) are true. Indeed, top-dimensional cycles on manifolds are constant on each component, so a square-summable cycle on an infinite component is constant 0. As a result, Conjecture 1.1 in dimension ≤ 2 follows from Poincaré duality.

I(3) is true. In [7], the authors show that **I(3)** is true for right-angled Coxeter groups. In [12], the author geometrizes arbitrary 3-dimensional Davis manifolds and shows that **I(3)** follows, Corollary 4.4, [12].

3.1 Singer’s Conjecture for Ruins

What follows are variations of **I(n)** for one-letter ruins, as defined in section 2.2.

II(n). *Let (W, S) be a Coxeter system whose nerve L is a triangulation of \mathbb{S}^{n-1} and let $t \in S$. Then $\mathcal{H}_i(\Omega(S, t), \partial\Omega(S, t)) = 0$ for $i > \frac{n}{2}$.*

V(n). *Let (W, S) be a Coxeter system whose nerve L is a triangulation of \mathbb{S}^{n-1} . Let $V \subseteq S$ and $t \in V$. Then $\mathcal{H}_i(\Omega(V, t), \partial\Omega(V, t)) = 0$ for $i > \frac{n}{2}$.*

Proposition 3.1.1. *II(n) implies that $\mathcal{H}_i(\partial\Omega(S, t)) = \mathcal{H}_i(\Omega(S, t))$ for $i > \frac{n}{2}$.*

Proof. Consider the long exact sequence of the pair $(\Omega, \partial\Omega) = (\Omega(S, t), \partial\Omega(S, t))$:

$$\dots \rightarrow \mathcal{H}_*(\partial\Omega) \rightarrow \mathcal{H}_*(\Omega) \rightarrow \mathcal{H}_*(\Omega, \partial\Omega) \rightarrow \dots$$

II(n) implies the third term vanishes, for $i > \frac{n}{2}$. The result follows from exactness. \square

The same proof applied to the long exact sequence of the pair $(\Omega(V, t), \partial\Omega(V, t))$ proves the following.

Proposition 3.2. $\mathbf{V}(n)$ implies that $\mathcal{H}_i(\partial\Omega(V, t)) = \mathcal{H}_i(\Omega(V, t))$ for $i > \frac{n}{2}$.

Singer's Conjecture for two-letter ruins. The following statement about two-letter ruins is needed for our program.

TR(n). Let (W, S) be a Coxeter system with nerve L a triangulation of \mathbb{S}^{n-1} . Let $V \subseteq S$ and let $T \subseteq V$ be a spherical subset with $\text{Card}(T) = 2$. Then $\mathcal{H}_i(\Omega(V, T), \partial\Omega(V, T)) = 0$ for $i > \frac{n}{2} + 1$.

3.3 Implications

Excision Isomorphisms. Now let $V \subseteq S$, be arbitrary; $T \subseteq V$ spherical, $\Omega := \Omega(V, T)$, $\partial\Omega := \partial\Omega(V, T)$. Recall that $\Sigma(V)$ is the subcomplex of Σ_{cc} consisting of cells of type T' , with $T' \subseteq V$. We have excision isomorphisms (as in [5]):

$$C_*(\Omega(V, T), \partial\Omega) \cong C_*(\Sigma(V), \widehat{\Omega}(V, T)), \quad (3.1)$$

and for any $s \in T$ and $T' := T - s$,

$$C_*(\Sigma(V - s), \widehat{\Omega}(V - s, T')) \cong C_*(\widehat{\Omega}(V, T), \widehat{\Omega}(V, T')). \quad (3.2)$$

Set $\widehat{\Omega} := \widehat{\Omega}(V, T)$, and $\widehat{\Omega}' := \widehat{\Omega}(V, T')$. Consider the long, weakly exact sequence of the triple $(\Sigma(V), \widehat{\Omega}, \widehat{\Omega}')$:

$$\dots \rightarrow \mathcal{H}_*(\widehat{\Omega}, \widehat{\Omega}') \rightarrow \mathcal{H}_*(\Sigma(V), \widehat{\Omega}') \rightarrow \mathcal{H}_*(\Sigma(V), \widehat{\Omega}) \rightarrow \dots$$

By equations (3.1) and (3.2), the left hand term excises to the homology of the $(V - s, T')$ -ruin, the right hand term to that of the (V, T) -ruin and the middle term to that of the (V, T') -ruin; leaving the sequence:

$$\dots \rightarrow \mathcal{H}_*(\Omega(V - s, T'), \partial) \rightarrow \mathcal{H}_*(\Omega(V, T'), \partial) \rightarrow \mathcal{H}_*(\Omega(V, T), \partial) \rightarrow \dots \quad (3.3)$$

Proposition 3.3.1. $[\mathbf{II}(n) \text{ and } \mathbf{TR}(n)] \implies \mathbf{V}(n)$.

Proof. It is clear that $\mathcal{H}_i(\Omega(V, t)) = 0$ for $i > \frac{n}{2}$ whenever $\text{Card}(V) \leq 2$, so we may assume that $\text{Card}(V) > 2$. We induct on $\text{Card}(S - V)$, **II**(n) giving us the base case. Let $V = V' \cup s$ and $t \in V'$. Assume the result holds for V . If

$m_{st} = \infty$ then $(\Omega(V', t), \partial) = (\Omega(V, t), \partial)$ and we are done. Otherwise, consider the sequence in equation (3.3), taking $T = \{s, t\}$, $T' = \{t\}$:

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{H}_n(\Omega(V', t), \partial) & \rightarrow & \mathcal{H}_n(\Omega(V, t), \partial) & \rightarrow & \mathcal{H}_n(\Omega(V, \{s, t\}), \partial) \\ & & \rightarrow & \mathcal{H}_{n-1}(\Omega(V', t), \partial) & \rightarrow & \mathcal{H}_{n-1}(\Omega(V, t), \partial) & \rightarrow \mathcal{H}_{n-1}(\Omega(V, \{s, t\}), \partial) \\ & & \vdots & & \vdots & & \vdots \\ & \rightarrow & \mathcal{H}_k(\Omega(V', t), \partial) & \rightarrow & \mathcal{H}_k(\Omega(V, t), \partial) & \rightarrow & \dots \end{array}$$

(where $k = \frac{n}{2} + 1$ if n is even, $k = \frac{n+1}{2}$ if n is odd). $\mathcal{H}_i(\Omega(V, t), \partial) = 0$ for $i > \frac{n}{2}$ by assumption and $\mathbf{TR}(n)$ implies $\mathcal{H}_i(\Omega(V, \{s, t\}), \partial) = 0$. So by exactness, $\mathcal{H}_i(\Omega(V', t), \partial) = 0$ for $i > \frac{n}{2}$. \square

Theorem 3.3.2. $\mathbf{V}(n) \implies \mathbf{I}(n)$.

Proof. Let $V \subseteq S$ and $t \in V$. Consider the following form of (3.3), where $T = \{t\}$:

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{H}_n(\Sigma(V - t)) & \rightarrow & \mathcal{H}_4(\Sigma(V)) & \rightarrow & \mathcal{H}_4(\Omega(V, t), \partial) \rightarrow \\ & & \rightarrow & \mathcal{H}_3(\Sigma(V - t)) & \rightarrow & \mathcal{H}_3(\Sigma(V)) & \rightarrow \mathcal{H}_3(\Omega(V, t), \partial) \rightarrow \\ & & \vdots & & \vdots & & \vdots \\ & \rightarrow & \mathcal{H}_k(\Sigma(V - t)) & \rightarrow & \mathcal{H}_k(\Sigma(V)) & \rightarrow & \mathcal{H}_k(\Omega(V, t), \partial) \rightarrow \end{array}$$

(where $k = \frac{n}{2} + 1$ if n is even, $k = \frac{n+1}{2}$ if n is odd). By $\mathbf{V}(n)$, $\mathcal{H}_i(\Omega(V, t), \partial) = 0$ for $i > \frac{n}{2}$. So by exactness,

$$\mathcal{H}_i(\Sigma(V - t)) \cong \mathcal{H}_i(\Sigma(V)),$$

for $i > \frac{n}{2}$. It follows that $\mathcal{H}_i(\Sigma) \cong \mathcal{H}_i(\Sigma(\emptyset)) = 0$ for $i > \frac{n}{2}$ and hence, by Poincaré duality, $\mathcal{H}_i(\Sigma) = 0$ for $i \neq \frac{n}{2}$. \square

4 Even Coxeter systems

4.1 The Combinatorics of Even systems

We present some of the background for the combinatorial arguments used in [11]. Let (W, S) be a Coxeter system. Given a subset U of S , define W_U to be the subgroup of W generated by the elements of U . (W_U, U) is a Coxeter system. A subset T of S is *spherical* if W_T is a finite subgroup of W . In this case, we will also say that the subgroup W_T is spherical. We say the Coxeter system (W, S) is *even* if for any $s, t \in S$ with $s \neq t$, m_{st} is either even or infinite.

Given $w \in W$, we call an expression $w = (s_1 s_2 \cdots s_n)$ *reduced* if there does not exist an integer $m < n$ with $w = (s'_1 s'_2 \cdots s'_m)$. Define the *length of w* , $l(w)$, to be the integer n such that $(s_1 s_2 \cdots s_n)$, is a reduced expression for w . Denote by $S(w)$ the set of elements of S which comprise a reduced expression for w . This set is well-defined, [4, Proposition 4.1.1].

For $T \subseteq S$ and $w \in W$, the coset wW_T contains a unique element of minimal length. This element is said to be (\emptyset, T) -reduced. Moreover, it is shown in [2,

Ex. 3, pp. 31-32], that an element is (\emptyset, T) -reduced if and only if $l(wt) > l(w)$ for all $t \in T$. Likewise, we can define the (T, \emptyset) -reduced elements to be those w such that $l(tw) > l(w)$ for all $t \in T$. So given $X, Y \subseteq S$, we say an element $w \in W$ is (X, Y) -reduced if it is both (X, \emptyset) -reduced and (\emptyset, Y) -reduced.

Shortening elements of W . We have the so-called “Exchange” **(E)** condition for Coxeter systems ([2, Ch 4. Section 1, Lemma 3] or [4, Theorem 3.3.4]):

- **(E)** Given a reduced expression $w = (s_1 \cdots s_k)$ and an element $s \in S$, either $\ell(sw) = k + 1$ or there is an index i such that

$$sw = (s_1 \cdots \widehat{s_i} \cdots s_k).$$

In the case of even Coxeter systems, the parity of a given generator in the set expressions for an element of W is well-defined. (We prove this herein, Lemma 4.3.4.) So, in **(E)**, $s_i = s$; i.e, if an element of $s \in S$ shortens a given element of W , it does so by deleting an instance of s in an expression for w .

It is also a fact about Coxeter groups ([4, Theorem 3.4.2]) that if two reduced expressions represent the same element, then one can be transformed into the other by replacing alternating subwords of the form $(sts \dots)$ of length m_{st} by the alternating word $(tst \dots)$ of length m_{st} . The proof of the first of the following two lemmas follows immediately from this. The proof of the second depends on the first Lemma 4.1.1 and may be found in [11].

Lemma 4.1.1. *Let $t \in S$, $w \in W_{S-t}$ and $v \in W$ with wtv reduced. If there exists an $r \in S(w) - S(v)$ with $(rt)^2 \neq 1$, then all r ’s appear to the left of all t ’s in any reduced expression for wtv .*

Lemma 4.2. *Let (W, S) be an even Coxeter system, let $t, s \in S$ be such that $2 < m_{st} < \infty$ and let $U_{st} = \{r \in S \mid m_{rt} = m_{rs} = 2\}$. Suppose that $tstw' = wtv$ (both reduced) where $w' \in W$, $w \in W_{S-t}$ and $S(v) \subset U_{st} \cup \{s, t\}$. Then $S(w) \subseteq U_{st} \cup \{s\}$.*

4.3 Coloring the Davis Complex

Here and for the remainder of this section, we require that (W, S) be an even Coxeter system with nerve L . Fix $t \in S$ and let $U := \{s \in S \mid m_{st} < \infty\}$, and let Ω and $\partial\Omega$ be defined as in Section 2. The following is a generalization of the argument put forth in [11].

Any $s \in U$ has the property that $m_{st} < \infty$. Let $S' := \{s \in U \mid m_{st} > 2\}$, and assume that S' is not empty. The group W_U has the following properties.

Lemma 4.3.1. *Suppose that L is flag. Then for $s, s' \in S'$, either $s = s'$, or $m_{ss'} = \infty$.*

Proof. Suppose that $s \neq s'$ and that $m_{ss'} < \infty$. Then $\{s, s'\} \in \mathcal{S}$, and since s, s' are both in U , the vertices corresponding to s, s' and t are pairwise connected

in L . L is a flag complex, so this implies that $\{s, s', t\} \in \mathcal{S}$. But

$$\frac{1}{m_{ss'}} + \frac{1}{m_{st}} + \frac{1}{m_{ts'}} \leq \frac{1}{m_{ss'}} + \frac{1}{4} + \frac{1}{4} \leq 1.$$

This contradicts $\{s, s', t\}$ being a spherical subset. So we must have that $m_{ss'} = \infty$. \square

Corollary 4.3.2. *Let $s \in S'$ and let $T \in \mathcal{S}_{\geq\{s,t\}}$. Then $m_{ut} = m_{us} = 2$ for $u \in T - \{s, t\}$.*

In other words, the generators from $T - \{s, t\}$ commute with both s and t .

Links. Now let L_{st} denote the link in L of the edge connecting the vertices s and t . The above Corollary states that the generators in the vertex set of L_{st} commute with both s and t . As in Lemma 4.2, denote this set of generators by U_{st} .

Of particular interest to us will be elements of W_U with a reduced expression of the form $tst \cdots st$ for some $s \in S'$. Since W is even, this expression is unique, and we have the following lemma.

Lemma 4.3.3. *Let $s \in S'$ and let $u \in W_{\{s,t\}}$ be such that $u = tst \cdots st$, is a reduced expression beginning and ending with t . Then u is $(U - t, U - t)$ -reduced.*

Lemma 4.3.4. *Let $V, T \subseteq S$ and consider the function $g_{VT} : W_V \rightarrow W_T$ induced by the following rule: $g_{VT}(s) = s$ if $s \in V \cap T$ and $g_{VT}(s) = e$ (the identity element of W) for $s \in V - T$. Then g_{VT} is a homomorphism.*

Proof. We show that g_{VT} respects the relations in W_V . Let $s, u \in V$ be such that $(su)^m = 1$. Then

$$g_{VT}((su)^m) = \begin{cases} (su)^m & \text{if } s \in T, u \in T \\ s^m & \text{if } s \in T, u \notin T \\ u^m & \text{if } u \in T, s \notin T \\ e & \text{if } s \notin T, u \notin T. \end{cases}$$

In all cases, since (W_V, V) is even, $g_{VT}((su)^m) = e$. \square

Group Action on Cosets. Then with $T \in \mathcal{S}_{\geq t}$ and U as above, we define an action of W_U on the set of cosets W_T/W_{T-t} : For $w \in W_U$ and $v \in W_T$, define

$$w \cdot vW_{T-t} = g_{UT}(w)vW_{T-t}. \quad (4.1)$$

Painting vertices of Ω . Set

$$A = \prod_{T \in \mathcal{S}_{\geq t}} W_T/W_{T-t}.$$

We call A the set of colors and note that it is a finite set. The action defined in equation (4.1) extends to a diagonal W_U -action on A ; for $w \in W_U$ and $a \in A$, write $w \cdot a$ to denote w acting on a . Let \bar{e} be the element of A defined by taking the trivial coset W_{T-t} for each $T \in \mathcal{S}_{\geq t}$. Vertices of Ω correspond to group elements of W_U , so we paint the vertices of Ω by defining a map $c : W_U \rightarrow A$ with the rule $c(w) := w \cdot \bar{e}$.

Remark 4.3.5. If an element $w \in W_U$ does not contain t in any reduced expression, then w acts trivially on the element \bar{e} , i.e. $w \cdot \bar{e} = \bar{e}$.

Painting boundary collars. We paint the space $wK(U)$ with $c(w)$. In this way, all of Ω is colored with some element of A . For vertices w and w' of the same component B of $\partial\Omega$, $h = w^{-1}w' \in W_{U-t}$, so $c(w') = c(wh) = wh \cdot \bar{e} = w \cdot \bar{e} = c(w)$, where the third equality follows from Remark 4.3.5. Therefore all of the boundary collar containing w is painted with $c(w)$. Note that each component of $\partial\Omega$ is monochromatic while the interior of Ω is not.

Lemma 4.3.6. *Let $D = B \times [0, 1]$ and $D' = B' \times [0, 1]$ be boundary collars where B and B' are different components of $\partial\Omega$. Suppose that the vertices of B and B' have the same color. Then $D \cap D' = \emptyset$.*

Proof. Suppose, by way of contradiction, that $D \cap D' \neq \emptyset$, i.e. there exist vertices $w \in B$, $w' \in B'$ such that $c(w) = c(w')$ and $wK(U) \cap w'K(U) \neq \emptyset$. Let $V = S(v)$, where $v = w^{-1}w'$, and since w and w' are from different components of $\partial\Omega$, $t \in V$. Now $c(w) = c(w') \Rightarrow w \cdot \bar{e} = wv \cdot \bar{e} \Rightarrow \bar{e} = v \cdot \bar{e}$. Thus, for any $T \in \mathcal{S}_{\geq t}$, we have that

$$v \cdot W_{T-t} = W_{T-t}. \quad (4.2)$$

But since $v \in W_V$, the action of v on W_V/W_{V-t} defined in (4.1) is left multiplication by v . But by equation (4.2), we have that $v \in W_{V-t}$; a contradiction. \square

c -collars. Now for $c \in A$, define the c -collar, F_c , to be the disjoint union of the boundary collars $D = B \times [0, 1]$ where each component B of $\partial\Omega$ has the color c . The collection of c -collars, for all colors c , is a finite cover of Ω .

4.4 Even and odd collars

Let $T = \{t\}$ and consider the homomorphism $g_{UT} : W_U \rightarrow W_t$ defined in Lemma 4.3.4. Under g_{UT} , an element $w \in W_U$ is sent to the identity in W_t if w has an even number of t 's present in some factorization (and therefore, all factorizations) as a product of generators from U and an element $w \in W_U$ is sent to $t \in W_t$ if w has an odd number of t 's present in factorizations. Thus, we call a vertex w *even* if $g_{UT}(w) = e$; *odd* if $g_{UT}(w) = t$. If two vertices w and w' are such that $c(w) = c(w')$, then clearly $g_{UT}(w) = g_{UT}(w')$, so we may also classify the colors as even or odd. A c -collar is even or odd as c is even or odd and we refer to it as an “even or odd collar.”

We will be employing a Mayer-Vietoris argument using the collars as individual pieces of the union. So, of fundamental importance will be how these

collars intersect. By Remark 4.3.5, we know that in order for the vertices of a Coxeter cell to support two different colors, this cell must be of type $T \in \mathcal{S}_{\geq t}$. But, for a cell to support two different *even* vertices, v and v' , this cell must be of type $T \in \mathcal{S}_{\geq \{s,t\}}$ for exactly one $s \in S'$ (uniqueness is given by Corollary 4.3.2). Moreover, $w = v^{-1}v'$ has the properties that (1) $\{s, t\} \subseteq S(w)$ and that (2) it contains at least two, and an even number of t 's in any factorization as a product of generators. Such a w we call *t-even*.

The intersection of even collars. Now let L be a flag triangulation of \mathbb{S}^{n-1} , so that Σ is an n -manifold. Let D_0 denote the boundary collar containing the vertex e . Fix $s \in S'$ and let D_2 denote the boundary collar containing the vertex u , where $u \in W_{\{s,t\}}$ is t -even and has a reduced expression ending in t . We study $D_0 \cap D_2$.

Lemma 4.4.1. *Let $W' := W_{U_{st}}$, where $U_{st} = \{r \in S \mid m_{rt} = m_{rs} = 2\}$, and let $K' = K(U) \cap uK(U)$. Denote by $W'K'$ the orbit of K' under W' . Then $D_0 \cap D_2 = W'K'$.*

Proof. For any $w \in W'$, the vertex w is in the same component of $\partial\Omega$ as e (by Remark 4.3.5), and therefore $wK(U) \subset D_0$. $wu = uw$, so wu is in the same component of $\partial\Omega$ as u and $wuK(U) \subset D_2$. Thus $wK' = wK(U) \cap wuK(U) \subset D_0 \cap D_2$.

Now let σ be a 0-simplex in $D_0 \cap D_2$. Then there exist $w, w' \in W_{U-t}$ such that $\sigma \in wK(U) \cap uw'K(U)$, i.e. σ is simultaneously the w - and uw' -translate of a 0-simplex σ' in $K(U)$. Let V be the spherical subset to which σ' corresponds and let $v \in W_V$ be such that $uw' = wv$. $c(e) = c(w)$ and $c(u) = c(uw')$, so w and uw' are differently colored even vertices of a Coxeter cell of type V . By the second paragraph of 4.4, $\{s', t\} \subseteq S(v) \subseteq V$ for exactly one $s' \in S'$ and v is t -even.

Claim 1: $s' = s$.

Pf: Since $w' \in W_{U-t}$, $c(u) = c(uw') = c(wv)$, i.e. u and wv act the same on every coordinate of \bar{e} . Consider the $\{s, t\}$ -coordinate. $u \in W_{\{s,t\}}$ is t -even, so $u \cdot W_s = uW_s$ and $uW_s \neq W_s$. But if $s \notin S(v)$, then v being t -even and $w \in W_{U-t}$ imply that $wv \cdot W_s = W_s$; which contradicts u and wv having the same color. So Claim 1 is true, and as a result $V \in \mathcal{S}_{\geq \{s,t\}}$ and $\sigma' \in K'$. Moreover, by Corollary 4.3.2, $V \subseteq U_{st} \cup \{s, t\}$. It remains to show that σ is in the W' -orbit of K' .

Claim 2: $S(w) \subseteq (U_{st} \cup \{s\})$.

Pf: Take a reduced expression for u which ends in t . If this expression begins with s , multiply u on the left by s , so that we have $suw' = swv$. The only change this can effect on $S(w)$ is either adding or subtracting an s , which is inconsequential to our claim. So, we may assume that u has a reduced expression of the form $tst \cdots st$ as described in Lemma 4.3.3. Hence, u is $(U-t, U-t)$ -reduced and uw' has a reduced expression beginning with the subword tst . wv has a reduced expression of the form $w''tv'$ where $w'' \in W_{U-t}$, $S(v') \subset U_{st} \cup \{s, t\}$ and where the difference between $S(w)$ and $S(w'')$ is contained in $U_{st} \cup \{s\}$. Claim 2 then follows from Lemma 4.2 applied to w'' .

We now finish the proof of Lemma 4.4.1. If $s \notin S(w)$, then $w \in W'$ and we are done since σ is the w -translate of σ' . If $s \in S(w)$, then w may be written as qs , with $q \in W'$ and since $s \in V$, $qsW_V = qW_V$. So σ is also the q -translate of σ' . \square

Proposition 4.4.2. $(D_0 \cap D_2) \cong \Sigma(W', U_{st})$, an infinite connected $(n-2)$ -manifold.

Proof. Since $S(u) = \{s, t\}$, K' is the geometric realization of the poset $\mathcal{S}_{\geq\{s,t\}} = \{V \in \mathcal{S} \mid \{s, t\} \subseteq V\}$. By Lemma 4.4.1, $(D_0 \cap D_2) \cong |W' \mathcal{S}_{\geq\{s,t\}}|$, and by Corollary 4.3.2, $\mathcal{S}_{\geq\{s,t\}}$ is isomorphic to $\mathcal{S}(U_{st})$ via the map $T \rightarrow T - \{s, t\}$. So $(D_0 \cap D_2) \cong |W' \mathcal{S}(U_{st})| = \Sigma(W', U_{st})$.

Recall that L_{st} denotes the link in L of the edge connecting s and t . Simplices in L_{st} correspond to spherical subsets $T \in \mathcal{S}$ such that neither s nor t is contained in T but $T \cup \{s, t\} \in \mathcal{S}$. So by Corollary 4.3.2, the vertex set of a simplex of L_{st} corresponds to a spherical subset of $\mathcal{S}(U_{st})$. Conversely, given a spherical subset $T \in \mathcal{S}(U_{st})$, $W_{T \cup \{s,t\}} = W_T \times W_{\{s,t\}}$, which is finite. So T corresponds to a simplex of L_{st} . Thus, L_{st} is the nerve of the Coxeter system (W', U_{st}) . Since L triangulates \mathbb{S}^{n-1} , L_{st} triangulates \mathbb{S}^{n-3} . It follows from Proposition 2.1 that $\Sigma(W', U_{st})$ is a contractible $(n-2)$ -manifold. \square

Corollary 4.4.3. Let $c, c' \in A$ be even. Then $(F_c \cap F_{c'})$ is a disjoint union of infinite $(n-2)$ -manifolds.

Proof. Suppose that $F_c \neq F_{c'}$ are both even collars and $F_c \cap F_{c'} \neq \emptyset$. Then there exist even vertices v and v' with $vK(U) \cap v'K(U) \neq \emptyset$. Let $w = v^{-1}v'$ and put $T = S(v^{-1}v')$. T is a spherical subset, and v and v' are both vertices of a cell of type T . So we have exactly one $s \in S'$ with $\{s, t\} \subseteq T$. Factor w as $w = xq$ where $x \in W_{\{s,t\}}$ is t -even and $q \in W_{T-\{s,t\}}$. Now, x may not have a reduced expression ending in t . If it does not, then xs does and it is in the same boundary collar as x and w . So let

$$u = \begin{cases} x & \text{if } x \text{ has a reduced expression ending in } t, \\ xs & \text{otherwise.} \end{cases}$$

Then $vK(U) \cap v'K(U) \subseteq vK(U) \cap vuK(U)$. Act on the left by v^{-1} and we are in the situation studied in Lemma 4.4.1 and Proposition 4.4.2. So $F_c \cap F_{c'}$ is the disjoint union of infinite connected 2-manifolds. \square

Remark 4.4.4. If W is right-angled, or if $S' = \emptyset$, then $W_U = W_{U-t} \times W_t$ and there is one even and one odd collar. This is why all the effort on the colors....because the ruins have branching points.

Multiple even collars. Suppose that $D_1, D_2, \dots, D_n, D_e$ are even boundary collars. Then

$$D_e \cap \left(\bigcup_{j=1}^n D_j \right) = (D_e \cap D_1) \cup \dots \cup (D_e \cap D_n),$$

and suppose that for some $1 \leq i < k \leq n$ we have that $(D_e \cap D_i)$ and $(D_e \cap D_k)$ are not disjoint. Let σ be a 0-simplex contained in $D_e \cap D_i \cap D_k$ corresponding to a coset of the form vW_T . Then there exists $w, w' \in W_T$ such that $v \in D_e$, $vw \in D_i$, $vw' \in D_k$ and $\sigma \in vK(U) \cap vwK(U) \cap vw'K(U)$. These three vertices are differently colored even vertices of a cell of type T , so $\{s, t\} \subseteq T$ for exactly one $s \in S'$ and both w and w' are t -even. Then, as in the proof of Corollary 4.4.3, it follows that $D_e \cap D_i = D_e \cap D_k \cong |W'S_{\geq\{s,t\}}|$. As a result, Corollary 4.4.3 generalizes to the following:

Corollary 4.4.5. *Let $F_{c_1}, F_{c_2}, \dots, F_{c_n}, F_{c_e}$ be even collars. Then*

$$\left(F_{c_e} \cap \left(\bigcup_{j=1}^n F_{c_j} \right) \right)$$

is a disjoint collection of infinite $(n-2)$ -manifolds.

Odd collars. We now consider how the odd collars intersect with the entire collection of even collars.

Lemma 4.4.6. *Define*

$$\partial_{in}(F_c) := \prod_{D \subset F_c} \partial_{in}(D).$$

Let \mathcal{F}_E denote the union of all even collars and let F_o be an odd collar, then $F_o \cap \mathcal{F}_E = \partial_{in}(F_o)$.

Proof. Since F_o is a disjoint union of boundary collars, it suffices to show that $D \cap \mathcal{F}_E = \partial_{in}(D)$ for some boundary collar $D \subset F_o$.

(\supseteq): Let σ be a 0-simplex in $\partial_{in}(D)$. Then σ corresponds to a coset of the form wW_V where $V \in \mathcal{S}_{\geq t}$ and $w \in W_U$ is an odd vertex of D . Consider the even vertex wt . Then since $t \in V$, $wW_V = wtW_V$, and $\sigma \in wtK(U) \subset \mathcal{F}_E$.

(\subseteq): Now suppose that σ is a 0-simplex contained in $D \cap \mathcal{F}_E$. Then there exists a spherical subset V and cosets $wW_V = w'W_V$ where w is odd and w' is even. Let $v = w^{-1}w'$. Since w is odd and w' is even, v must contain an odd number of t 's in any of its reduced expressions. Therefore $t \in V$ and $\sigma \in \partial_{in}(D)$. \square

As before, let \mathcal{F}_E denote the union of all even collars, and now let \mathcal{F}_O denote the union of a sub-collection of odd collars. Let $\mathcal{F}_{E'} = \mathcal{F}_E \cup \mathcal{F}_O$ and let F_o be an odd collar not included in \mathcal{F}_O . Then by Lemma 4.4.6,

$$F_o \cap \mathcal{F}_{E'} = (F_o \cap \mathcal{F}_E) \bigcup (F_o \cap \mathcal{F}_O) = \partial_{in}(F_o) \bigcup (F_o \cap \mathcal{F}_O).$$

Any 0-simplex in F_o which is also in a different collar must be of the form wW_V , where w is a vertex of F_o and $V \in \mathcal{S}_{\geq t}$. Therefore $(F_o \cap \mathcal{F}_O) \subset \partial_{in}(F_o)$ and $F_o \cap \mathcal{F}_{E'} = \partial_{in}(F_o)$.

It is clear from the product structure on boundary collars that $\partial_{in}(F_o) \cong F_o \cap \partial\Omega$, the latter a disjoint collection of components of $\partial\Omega$. Since L is flag, we have a 1-1 correspondence between Coxeter cells of any component of $\partial\Omega$ and cells of $\Sigma(W_{U-t}, U-t)_{cc}$. Denote by L_t the link in L of the vertex corresponding to t , it is a triangulation of \mathbb{S}^{n-2} and it is isomorphic to the nerve of $(W_{U-t}, U-t)$. So we have the following corollary.

Corollary 4.4.7. *Let $\mathcal{F}_{E'}$ and F_o be as above. Then $F_o \cap \mathcal{F}_{E'}$ is a disjoint collection of $(n-1)$ -manifolds.*

Example 4.4.8. The following is representative of our situation. Suppose $L = \mathbb{S}^1$, and $U = \{t, r, s \mid (rt)^2 = 1, (st)^4 = 1\}$. Ω is represented in Figure 1. The black dots represent the vertices of the Coxeter cellulation, with the vertices e and tst labeled. The even collars are shaded. Even boundary collars intersect in a 0-simplex corresponding to the spherical subset $\{s, t\}$. The intersection of one odd collar and all evens is the inner boundary of the odd collar.

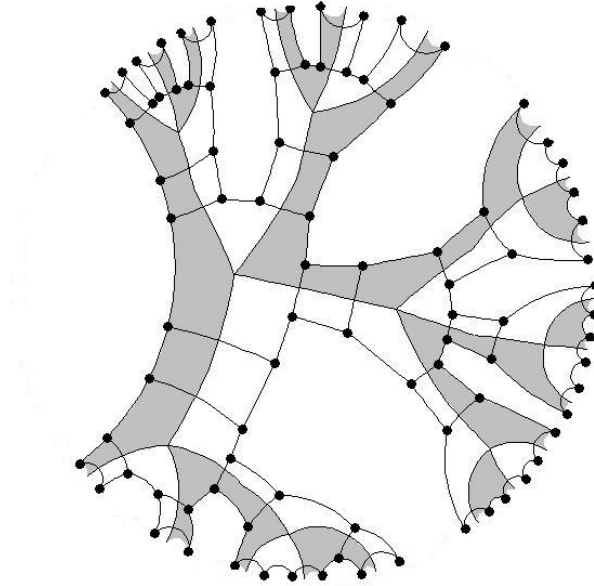


Figure 1: Even and Odd Colors of Ω

4.5 Inductive arguments in the case (W, S) is even

Consider the following restatements of conjectures **I**(n), **II**(n) and **V**(n), each in the case that (W, S) is *even* and L is a *flag* triangulation of \mathbb{S}^{n-1} . (Here the “E” stands for even, the “F” for flag.)

EFI(n). Let (W, S) be an even Coxeter system whose nerve L is a flag triangulation of \mathbb{S}^{n-1} . Then

$$\mathcal{H}_i(\Sigma) = 0 \text{ for } i \neq \frac{n}{2}.$$

EFII(n). Let (W, S) be an even Coxeter system whose nerve L is a flag triangulation of \mathbb{S}^{n-1} and let $t \in S$. Then $\mathcal{H}_i(\Omega(S, t), \partial\Omega(S, t)) = 0$ for $i > \frac{n}{2}$.

EFV(n). Let (W, S) be an even Coxeter system whose nerve L is a flag triangulation of \mathbb{S}^{n-1} . Let $V \subseteq S$ and $t \in V$. Then $\mathcal{H}_i(\Omega(V, t), \partial\Omega(V, t)) = 0$ for $i > \frac{n}{2}$.

EFTR(n). Let (W, S) be an even Coxeter system with nerve L a flag triangulation of \mathbb{S}^{n-1} . Let $V \subseteq S$ and let $T \subseteq V$ be a spherical subset with $\text{Card}(T) = 2$. Then $\mathcal{H}_i(\Omega(V, T), \partial\Omega(V, T)) = 0$ for $i > \frac{n}{2} + 1$.

A version of **EFTR**(4) is proven in [11], which only requires showing the top dimensional ℓ^2 -homology vanishes and the proof of which only requires the nerve L being flag. So, the proof given in [11] does generalize to the following “top-dimensional only” version of **EFTR**(n). Note that the following shows that **EFTR**(3) is true, and in fact, we can also drop the even hypothesis.

Proposition 4.5.1. Let $n \geq 3$ and (W, S) be a Coxeter system whose nerve L is flag triangulation of \mathbb{S}^{n-1} . Then $\mathcal{H}_n(\Omega(V, T), \partial\Omega(V, T)) = 0$.

Proof. If $\mathcal{S}(V)_{>T}^{(n)} = \emptyset$, then $\Omega(V, T)$ does not contain n -dimensional cells, and we are done. So assume that $\mathcal{S}(V)_{>T}^{(n)} \neq \emptyset$. The codimension 1 faces of n -cells of $\Omega(V, T)$ are either faces of one other n -cell in $\Omega(V, T)$ (Σ is an n -manifold), or they are free faces, i.e they are not faces of any other n -cell in $\Omega(V, T)$.

Suppose that cells of type $T' \in \mathcal{S}(V)_{>T}^{(n)}$ have a co-dimension one face of type R which is a face of another n -cell in $\Omega(V, T)$ of type T'' . Then any relative n -cycle must be constant on adjacent cells of type T' and T'' , where $T' = R \cup \{r\}$, and $T'' = R \cup \{s\}$, $R \in \mathcal{S}(V)_{>T}^{(n-1)}$ and $r, s \in V$. Since L is flag and $(n-1)$ -dimensional, $m_{rs} = \infty$. So in this case, there is a sequence of adjacent n -cells with vertex sets $W_{T'}, W_{T''}, sW_{T'}, srW_{T''}, sr sW_{T'}, sr srW_{T''}, \dots$. Hence, this constant must be 0.

Now suppose that for a given n -cell of $\Omega(V, T)$, every co-dimension one face is free. This cell has faces not contained in $\partial\Omega(V, T)$, so relative n -cycles cannot be supported on this cell. \square

Remark 4.5.2. Note the importance of **EFTR**(n) to this program, as you see in Theorem 4.6.5 below. If a generalized version of this could be proved, then the program would move forward to prove further cases of Conjecture 1.3.

4.6 Inductive Arguments

We now generalize the steps used in [11], presenting inductive arguments on the painted Davis Complex as a partially successful program to prove **EFI**(n). Since the proofs of Proposition 3.3.1 and Theorem 3.3.2 do not depend on the even nor odd hypotheses, the same proofs give us the following.

4.6.1

$[\mathbf{EFII}(n) \text{ and } \mathbf{EFTR}(n)] \implies \mathbf{EFV}(n)$.

4.6.2

$\mathbf{EFV}(n) \implies \mathbf{EFI}(n)$.

Proposition 4.6.3. For $k \in \mathbb{Z}$, $[\mathbf{EFI}(2k-2) \text{ and } \mathbf{EFI}(2k-1)] \implies \mathbf{EFII}(2k)$.

Proof. It suffices to calculate $\mathcal{H}_*(\Omega, \partial\Omega)$. We first show that $\mathcal{H}_n(\Omega, \partial\Omega) = 0$. Consider the long exact sequence of the pair $(\Omega, \partial\Omega)$:

$$\rightarrow \mathcal{H}_n(\Omega) \rightarrow \mathcal{H}_n(\Omega, \partial\Omega) \rightarrow \mathcal{H}_{n-1}(\partial\Omega) \rightarrow$$

Ω is an n -dimensional manifold with infinite boundary, so $\mathcal{H}_n(\Omega) = 0$ and $\mathcal{H}_{n-1}(\partial\Omega) = 0$. Then by exactness, $\mathcal{H}_n(\Omega, \partial\Omega) = 0$.

Now, let $i > \frac{n}{2}$ and let $\mathcal{F}_{E'}$ denote the union of a collection of even collars or the union of all evens and a collection of odd collars. Let F_c be a collar not contained in $\mathcal{F}_{E'}$ where if $\mathcal{F}_{E'}$ is not all the even collars, require that F_c be an even collar. Let $\partial_{E'} = \mathcal{F}_{E'} \cap \partial\Omega$ and let $\partial_{F_c} = F_c \cap \partial\Omega$. Note that $\partial_{E'} \cap \partial_{F_c} = \emptyset$ and consider the relative Mayer-Vietoris sequence of the pair $(\mathcal{F}_{E'} \cup F_c, \partial_{E'} \cup \partial_{F_c})$:

$$\dots \rightarrow \mathcal{H}_i(\mathcal{F}_{E'}, \partial_{E'}) \oplus \mathcal{H}_i(F_c, \partial_{F_c}) \rightarrow \mathcal{H}_i(\mathcal{F}_{E'} \cup F_c, \partial_{E'} \cup \partial_{F_c}) \rightarrow \mathcal{H}_{i-1}(\mathcal{F}_{E'} \cap F_c) \rightarrow \dots$$

Assume that $\mathcal{H}_{i-1}(\mathcal{F}_{E'}, \partial_{E'}) = 0$. Each collar retracts onto its boundary, so $\mathcal{H}_i(F_c, \partial_{F_c}) = 0$. If F_c is even, then the last term vanishes by Corollary 4.4.3 and $\mathbf{EFI}(n-2)$ and since $i-1 > \frac{n-2}{2}$, if F_c is odd, then the last term vanishes by 4.4.7, $\mathbf{EFI}(n-1)$ and since for n even, $i > \frac{n}{2}$ implies $i-1 > \frac{n-1}{2}$. In either case, exactness implies that $\mathcal{H}_i(\mathcal{F}_{E'} \cup F_c, \partial_{E'} \cup \partial_{F_c}) = 0$. It follows from induction that $\mathcal{H}_3(\Omega, \partial\Omega) = 0$. \square

Remark 4.6.4. Note that if n is odd, then with these hypotheses we are unable to guarantee the vanishing of the $\mathcal{H}_{(n-1)/2}(\mathcal{F}_{E'} \cap F_c)$ term for odd colors F_c . However, if we knew the inclusion map of the intersection of the painted boundary collars into the direct sum in the Mayer-Vietoris sequence was injective, we wouldn't need both dimensional statements. Since this "even-flag" argument is pretty technical, I am unsure of the most general statement that can be made.

It is known that $\mathbf{I}(2)$ and $\mathbf{I}(3)$ are true and therefore the more specific statements $\mathbf{EFI}(2)$ and $\mathbf{EFII}(3)$ are true. The purpose of [11] is to prove that $\mathbf{EFI}(4)$ is true. This is done in a manner exactly like that spelled out above, including the fact that $\mathbf{EFTR}(4)$ is true. Thus the main result of [11] is generalized by the following statement.

Theorem 4.6.5. $[\mathbf{EFI}(2k-2), \mathbf{EFI}(2k-1) \text{ and } \mathbf{EFTR}(2k)] \implies \mathbf{EFI}(2k)$.

Proof. By Proposition 4.6.3, the first two hypotheses give us that know that $\mathbf{EFII}(2k)$ is true. Then, along with $\mathbf{EFTR}(2k)$, this implies that $\mathbf{EFV}(2k)$ is true (see 4.6.1). Finally, by 4.6.2, we can conclude that $\mathbf{EFI}(2k)$ is true. \square

References

- [1] E. M. Andreev. On convex polyhedra of finite volume in Lobačevskii space. *Math. USSR Sbornik*, 12(2):255–259, 1970.
- [2] N. Bourbaki. *Lie Groups and Lie Algebras, Chapters 4-6*. Springer-Verlag, Berlin, Heidelberg and New York, 2002.
- [3] M. W. Davis. Groups generated by reflections and aspherical manifolds not covered by Euclidean space. *Annals of Mathematics*, 117:293–294, 1983.
- [4] M. W. Davis. *The Geometry and Topology of Coxeter Groups*. Princeton University Press, Princeton, 2007.
- [5] M. W. Davis, J. Dymara, T. Januszkiewicz, and B. Okun. Weighted L^2 -cohomology of Coxeter groups. *Geometry & Topology*, 11:47–138, 2007.
- [6] M. W. Davis and G. Moussong. Notes on nonpositively curved polyhedra. Ohio State Mathematical Research Institute Preprints, 1999.
- [7] M. W. Davis and B. Okun. Vanishing theorems and conjectures for the ℓ^2 -homology of right-angled Coxeter groups. *Geometry & Topology*, 5:7–74, 2001.
- [8] B. Eckmann. Introduction to ℓ^2 -methods in topology: reduced ℓ^2 -homology, harmonic chains, ℓ^2 -beti numbers. *Israel Journal of Mathematics*, 117:183–219, 2000.
- [9] J. Lott and W. Lück. ℓ^2 -topological invariants of 3-manifolds. *Invent. Math.*, 120:15–60, 1995.
- [10] G. Moussong. *Hyperbolic Coxeter Groups*. PhD thesis, The Ohio State University, 1998.
- [11] T. A. Schroeder. The ℓ^2 -homology of even Coxeter groups. *Algebraic & Geometric Topology*, 9(2):1089–1104, 2009. DOI number: 10.2140/agt.2009.9.1089.
- [12] T. A. Schroeder. Geometrization of 3-dimensional Coxeter orbifolds and Singer’s conjecture. *Geometriae Dedicata*, 140(1):163ff, 2009. DOI number: 10.1007/s10711-008-9314-5.